1 Sheet 3

1.1 Excercise 1

Recall that a group G is called **abelian** if $gh = hg$ for all $g, h \in G$, or in other words, if $G = Z(G)$.

(a) Show that if a Lie group G is abelian, then its Lie algebra $\mathfrak{g}$ is also abelian.

(b) Show that the converse also holds if G is connected.

Solution. (a) Since the Lie group G is abelian, $\mathrm{Ad}_g = \mathrm{id}_G$ for all $g \in G$. Then the adjoint representation (of the group) of G sends every element $g \in G$ to the identity in $\mathrm{GL}(\mathfrak{g})$. This means that the adjoint representation (of the algebra) $X \mapsto \mathrm{ad}_X$ is the differential of a constant map, so it is zero.

(b) If the Lie group G is connected, then it is generated by the image of the exponential map. The claim then follows from Baker–Campbell–Hausdorff's theorem. Alternatively, we can avoid using the Baker–Campbell–Hausdorff's theorem by "reversing" the proof of (a). Precisely, if the Lie algebra $\mathfrak{g}$ is abelian, then Then the adjoint representation of the group has the same differential as a constant map from G to $\mathrm{GL}(\mathfrak{g})$. Since G is connected, we conclude by Theorem 4 (a) in the notes that the map $G \to \mathrm{GL}(\mathfrak{g}), g \mapsto \mathrm{Ad}_{g,*}$ is constant. In other words, we have $\mathrm{Ad}_{g,*}(X) = AX$ for all $X \in \mathfrak{g}$ and $g \in G$. In particular $\mathrm{Ad}_{e,*}(X) = X$, so that $A = 1_n$. Applying Theorem 4 (a) again, we conclude that Ad_g itself is the identity map of G.

1.2 Excercise 2

Show that if a Lie group G is abelian and connected, then the exponential map

$$\exp : \mathfrak{g} \to G$$

is surjective. Conclude that $G = \mathfrak{g}/\Gamma$ for a discrete additive subgroup $\Gamma \subset \mathfrak{g}$.]

(above, ***discrete*** *refers to the topological property, i.e. in which any point is its own open subset; we say that X is a discrete subset of a topological space M if the latter can be covered by open subsets which intersect X in either zero or one point).*

Solution. The first statement follows from the fact that the image of the exponential map $\exp : \mathfrak{g} \to G$ generates G as a Lie group, and Baker–Campbell–Hausdorff's theorem.
Now consider the exponential map as a surjective homomorphism of Lie groups. Then its kernel Γ is an additive subgroup of $\mathfrak{g}$, and $G = \mathfrak{g}/\Gamma$ holds as an isomorphism of Lie groups. Finally we prove that $\Gamma \subset \mathfrak{g}$ is discrete. Let $U \subset \mathfrak{g}$ be an open neighborhood of zero, such that the restriction

$$\exp\colon U \to \exp(U)$$

is a diffeomorphism. In particular, $\ker(\exp) \cap U = \{0\}$, so that the singleton $\{0\}$ is open in $\ker(\exp)$. We conclude by applying left multiplication.

1.3 Excercise 3

The standard example of the situation in the previous problem is $\mathfrak{g} = \mathbb{R}$ and $G = S_1$. What about in the complex case, what are the possible complex abelian Lie groups G with $\mathfrak{g} = \mathbb{C}$?

Solution. These Lie groups are quotients $G = \mathbb{C}/\Gamma$, where Γ is a lattice. If Γ has rank 1, then $G = \mathbb{C}^*$. If Γ has rank 1, then G is a one dimensional complex torus (that is, a riemann surface of genus one, endowed with a group structure). It turns out that

$$\mathbb{C}/\Gamma_1 \cong \mathbb{C}/\Gamma_2$$

if and only if there is a complex number $\gamma \in \mathbb{C}^*$ such that $\Gamma_1 = \gamma\Gamma_2$. By using this, one can see that each isomorphism class of these groups contains a unique representative of the form $\mathbb{C}/\langle 1, \tau\rangle$, where $\langle 1, \tau\rangle$ is the lattice generated by 1 and τ, and τ falls in the region

$$\{\tau \in \mathbb{C} \mid |Im(\tau)| > 0, |Re(\tau)| \leq \frac{1}{2},\ |\tau| \geq 1\}.$$

1.4 Excercise 4

Recall that a topological space M is called **simply connected** if it is

- (path)-connected, i.e. any two points of M can be joined by a (continuous) path $\gamma : [0,1] \to M$
- any two paths $\gamma, \gamma' : [0,1] \to M$ with the same start and end points $\gamma(0) = \gamma'(0), \gamma(1) = \gamma'(1)$ can be related by a (continuous) **homotopy**

  $$H : [0,1] \times [0,1] \to M$$

  such that $H(t,0) = \gamma(t)$, $H(t,1) = \gamma'(t)$ and $H(0,t) = \gamma(0) = \gamma'(0)$, $H(1,t) = \gamma(1) = \gamma'(1)$ for all t (intutively, we think of $H_t = H(-,t)$ as giving us a family of paths that connect the paths γ and γ', all the while keeping the endpoints fixed).

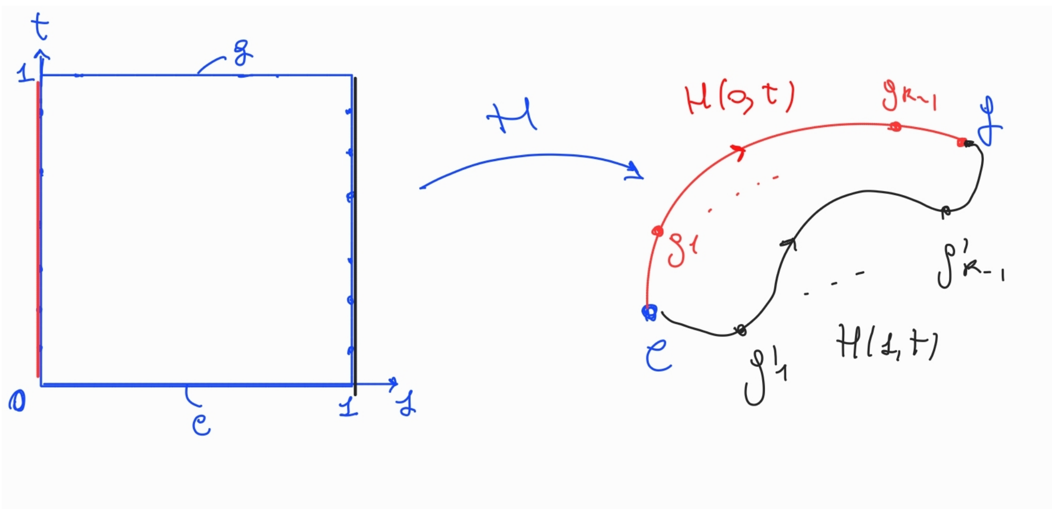
Excerise 4. Work out the details in Theorem 4.(b) in class.

Solution. [1] We have an homotopy

$$H : [0,1] \times [0,1] \to M$$

such that

- $H(t,0)$ is a path through the $g_i' s$,
- $H(t,1)$ is a path through the $g_i' s$,
- $H(s,0) = e$ and $H(s,1) = g$.

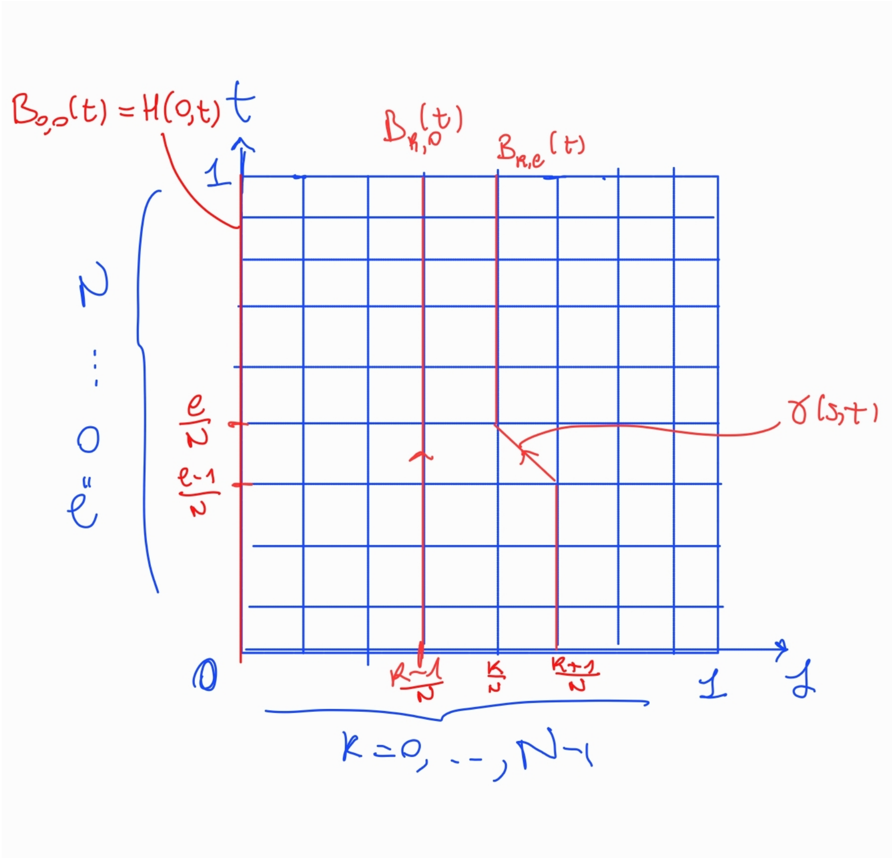


[1]Take a look at Chapter 3 of [?] for further details.

One can prove that there exists a positive integer N such that

$$H(s,t)H(s',t')^{-1} \in U \text{ for all } |s-s'| < 2/N,\ |t-t'| < 2/N. \tag{1}$$

Then we decompose $[0,1]\times[0,1]$ in a grid as in the picture. We have points k/N on the s-axis, for $k = 0,\dots,N-1$, and we have points l/N on the t-axis, for $l = 0,\dots,N$. In this way, every square in the grid has edge of $1/N$.



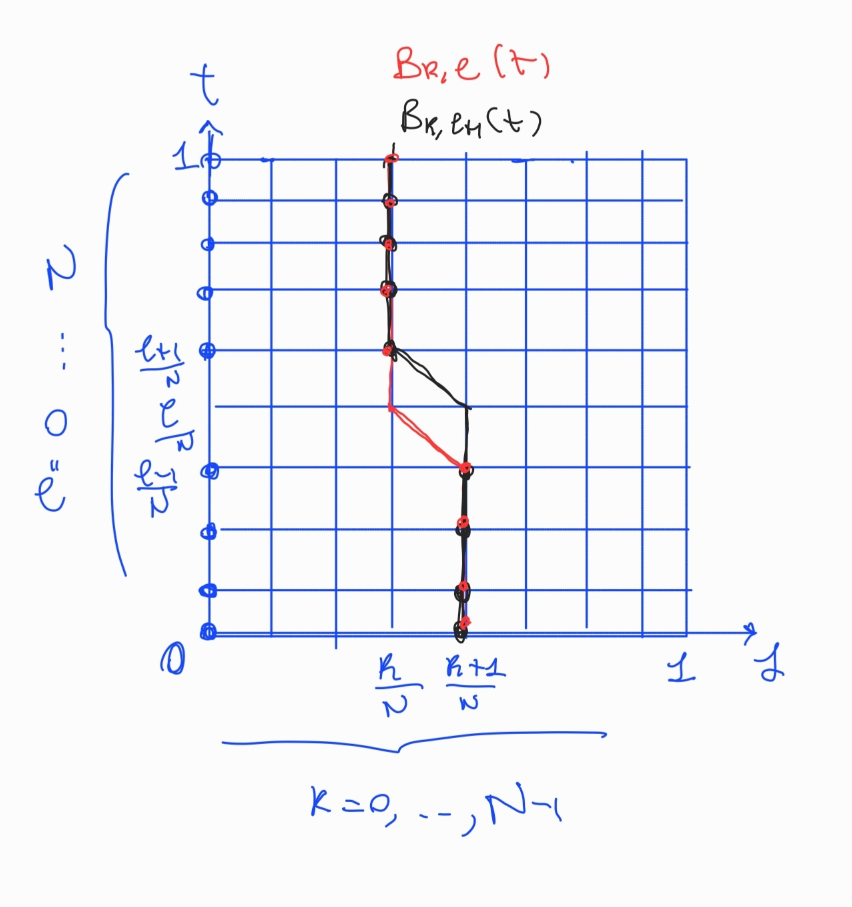
Then we define paths

$$B_{k,l} = \begin{cases} H\left(\frac{k+1}{N}, t\right) & \text{if } 0 \le t < \frac{l-1}{N} \\ H\left(\gamma(s,t)\right) & \text{if } \frac{l-1}{N} \le t < \frac{l}{N} \\ H\left(\frac{k}{N}, t\right) & \text{if } \frac{l-1}{N} \le t \le 1, \end{cases} \tag{2}$$

where $\gamma(s,t)$ is represented in the picture above . Order this sequence of paths as follows

$$\begin{aligned} H(0,t) = B_{0,0}(t), B_{0,1}(t), B_{0,2}(t), \dots, B_{0,N}(t), \\ B_{1,0}(t), B_{1,1}(t), \dots\dots\dots, B_{1,N}(t), \\ \dots\dots\dots, \\ \dots\dots\dots, B_{N-1,N}(t), H(1,t). \end{aligned}$$

The paths $B_{k,l}(t)$ coincides with $B_{k,l+1}(t)$ for $t = 0, \frac{1}{N}, \dots, \frac{l-1}{N}, \frac{l+1}{N}, \dots, 1$. The same is true for $B_{k,N}$ and $B_{k+1.0}$.



Then

$$F(B_{k,l}) = F\left(B_{k,l}(0)B_{k,l}\left(\frac{1}{N}\right)^{-1}\right)\cdots F\left(B_{k,l}\left(\frac{l-1}{N}\right)B_{k,l}\left(\frac{l+1}{N}\right)^{-1}\right)\cdots F\left(B_{k,l}\left(\frac{N-1}{N}\right)B_{k,l}\left(1\right)^{-1}\right) \tag{3}$$

is well defined because of (1), and it is constant along the above sequence of paths. This completes the proof modulo verifying independence of our construction from the chosen partition of $[0,1]\times[0,1]$.